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BOUNDARY LAYERS IN FREE CONVECTION

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The problem of free convection and mass transfer near a vertical wall is studied for the cases where the motion is described by the classical Oberbeck–Boussinesq model and the model of microconvection. In both cases, boundary layers are developed at high Schmidt numbers. Formulas for Nusselt (local and overall) numbers are obtained by solving the relevant problems for these layers. Initial asymptotic forms are also considered.

Introduction. We consider the problem of free convection in a viscous, incompressible fluid near a vertical wall (substrate) and transfer of an admixture in the case where the density of the solution depends on the admixture concentration. The fluid flow that arises in this case is called free or natural convection. Free convection has been much studied by various methods, including the boundary-layer method (see, e.g., [1–5]). In the present paper, we consider the case where the kinematic viscosity ν and the diffusivity D satisfy the condition $D \ll \nu$. At high Schmidt numbers $\mathrm{Sc} = \nu/D$, a dynamic-diffusion layer with thickness of the order of $(\mathrm{Re}^2\,\mathrm{Sc})^{-1/4}$ can be distinguished in the flow region. Outside this layer, the admixture concentration differs only slightly from the average, and the flow regime depends on the Reynolds number Re as follows. For $\mathrm{Re} \ll \mathrm{Sc}^{1/2}$, the flow corresponds to the Stokes approximation, and for $\mathrm{Re} \sim \mathrm{Sc}^{1/2}$, the flow is described by steady Navier–Stokes equations. For $\mathrm{Re} \gg \mathrm{Sc}^{1/2}$, there is also a purely dynamic layer with thickness of the order of $(\mathrm{Sc}/\mathrm{Re}^2)^{1/4}$, which is adjacent to the dynamic-diffusion layer at the inner edge and the state of rest at the outer edge.

Self-similar solutions are used to obtain formulas for Nusselt numbers Nu (local and overall) that are similar to the formulas obtained in [3] from an analysis of the self-similar solutions for an ordinary dynamic-diffusion boundary layer [2]. These formulas are also applicable to flows at moderate and low Reynolds numbers. In addition, the proposed approach allows one to distinguish a dynamic-diffusion boundary layer in the case of microconvection, where the Oberbeck–Boussinesq model is inapplicable. The model of microconvection that can be used instead was developed by Pukhnachev [6]. A similar approach was used by Perera and Sekerka [7] to study concentration convection. A comparative analysis of the velocity and concentration (temperature) fields calculated using both models was performed by Goncharova [8, 9]. The possibility of distinguishing a dynamic-diffusion boundary layer in microconvection (an ordinary boundary layer is not distinguished) allows one to compare integral flow characteristics, such as Nusselt numbers, for these models. For both models, initial asymptotic forms of the process are considered and formulas for Nu are obtained.

The method proposed can also be applied in the case of free convection near a vertical wall due to a nonuniform distribution of the fluid temperature. In this case, the Prandtl number is used instead of the Schmidt number, and concentration is replaced by temperature. However, situations in which the thermal conductivity is much less than the viscosity are rare. At the same time, the case where $D \ll \nu$ is typical. For example, for growth of thin films from a solution–melt of semiconducting materials, ν is of the order of 10^{-2}–10^{-3} cm^2/sec and $D \sim 10^{-5}$ cm^2/sec.

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We consider the problem of determining the u and v components of the velocity vector $\boldsymbol{v}$, the concentration c, and the difference between the pressure and the hydrostatic pressure p in the region $y > 0$ bounded by an infinite vertical wall $\{y = 0\}$. The gravity force is directed along the Ox axis. In the (x, y) coordinates, the acceleration of gravity has the form $\boldsymbol{g} = (-g, 0)$. We assume that the density of the melt ρ is a linear function of the concentration: $\rho = \rho_0[1 + \beta(c - c_0)]$. Here ρ_0 and c_0 are the average density and concentration of the solution and $\beta = (1/\rho_0)\, d\rho/dc = \text{const}$ (for definiteness, we set $\beta > 0$). Then, the equations of motion in the Boussinesq approximation have the form

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) - g\beta(c - c_0); \tag{1}$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\Big); \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \tag{3}$$

$$\frac{\partial c}{\partial t} + u\,\frac{\partial c}{\partial x} + v\,\frac{\partial c}{\partial y} = D\Big(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\Big). \tag{4}$$

At the initial moment, we assume equilibrium

$$u\Big|_{t=0} = v\Big|_{t=0} = 0, \qquad c\Big|_{t=0} = c_0, \tag{5}$$

and the boundary conditions have the form

$$u\Big|_{y=0} = v\Big|_{y=0} = 0, \qquad c\Big|_{y=0} = c_*, \qquad c \underset{y\to\infty}{\longrightarrow} c_0, \tag{6}$$

where $c_* = \text{const} > 0$.

The microconvection problem is to find the concentration c, the modified velocity $\boldsymbol{w} = \boldsymbol{v} + \beta D\nabla c$, and the modified pressure $q = p/\rho_* - gx + \beta(\nu - D)D\Delta c$, where $\rho = \rho_*(1 - \beta(c - c_0))^{-1}$, that satisfy the following initial-boundary-value problem:

$$\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{w}\nabla\boldsymbol{w} - \beta D(\nabla c\nabla\boldsymbol{w} - \nabla\boldsymbol{w}\nabla c) + \beta^2 D^2(\Delta c\nabla c - \nabla|\nabla c|^2/2)$$

$$= (1 - \beta(c - c_0))(-\nabla q + \nu\Delta\boldsymbol{w}) + \beta(c - c_0)\boldsymbol{g}, \tag{7}$$

$$\operatorname{div}\boldsymbol{w} = 0, \quad \frac{\partial c}{\partial t} + \boldsymbol{w}\nabla c - \beta D|\nabla c|^2 = D(1 - \beta(c - c_0))\Delta c;$$

$$w_1\Big|_{t=0} = w_2\Big|_{t=0} = 0, \qquad c\Big|_{t=0} = c_0; \tag{8}$$

$$w_1\Big|_{y=0} = 0, \qquad w_2\Big|_{y=0} = \beta D\,\frac{\partial c}{\partial y}\Big|_{y=0}, \qquad c\Big|_{y=0} = c_*, \qquad c \underset{y\to\infty}{\longrightarrow} c_0. \tag{9}$$

Here w_1 and w_2 are the components of the vector $\boldsymbol{w}$.

Boundary Layers in Steady Flow. Boundary conditions (6) and (9) in the problems considered above do not depend on time. Therefore, after decay of the initial perturbations, the process becomes steady, so that one can seek time-independent solutions of problems (1)–(6) and (7)–(8) ignoring conditions (5) and (8). These solutions describe the longest and most important part of the entire process.

We note that system (1)–(4) has a solution of the form

$$u = v = 0, \qquad p = 0, \qquad c = c_0, \tag{10}$$

that satisfies all boundary conditions (6), except for the constraint on the concentration at $y = 0$. To resolve this discrepancy, we determine the asymptotic forms of this system in the limit $\text{Sc} \to \infty$, treating (10) as the external solution. We assume that the problem has a certain length scale l, e.g., the dimension of the substrate. We define the velocity scale by the formula $U = \sqrt{g\beta l(c_0 - c_*)}$. It is known [1] that problem (1)–(6) depends on two dimensionless similarity criteria: the Schmidt number defined above and the Reynolds number

$$\mathrm{Re} = Ul/\nu = \left[g\beta(c_0 - c_*)l^3/\nu^2\right]^{1/2}. \tag{11}$$

Below, we assume that

$$\mathrm{Re}^2\,\mathrm{Sc} \to \infty \quad \text{as} \quad \mathrm{Sc} \to \infty. \tag{12}$$

which is equivalent to small νD. Within the framework of this assumption, no restrictions are imposed on the Reynolds numbers.

Let

$$x \sim l, \qquad y \sim l(\mathrm{Sc}\,\mathrm{Re}^2)^{-1/4}, \qquad u \sim U\,\mathrm{Sc}^{-1/2}, \qquad v \sim U(\mathrm{Sc}^3\,\mathrm{Re}^2)^{-1/4},$$

$$p \sim \rho_0 U^2(\mathrm{Sc}\,\mathrm{Re}^2)^{-1/2}, \qquad c \sim c_*.$$

Then, in Eq. (1) we have the following orders of magnitude:

$$u\,\frac{\partial u}{\partial x}, \quad v\,\frac{\partial u}{\partial y} \sim \frac{U^2}{l}\,\mathrm{Sc}^{-1}; \qquad \frac{1}{\rho_0}\,\frac{\partial p}{\partial x} \sim \frac{U^2}{l}\,(\mathrm{Sc}\,\mathrm{Re}^2)^{-1/2};$$

$$\nu\,\frac{\partial^2 u}{\partial x^2} \sim \frac{\nu U}{l^2}\,\mathrm{Sc}^{-1/2}; \qquad \nu\,\frac{\partial^2 u}{\partial y^2} \sim \frac{\nu U}{l^2}\mathrm{Re}; \qquad g\beta(c - c_0) \sim \frac{U^2}{l}.$$

In Eq. (2), we have

$$u\,\frac{\partial v}{\partial x}, \quad v\,\frac{\partial v}{\partial y} \sim \frac{U^2}{l}\,(\mathrm{Sc}^5\,\mathrm{Re}^2)^{-1/4}; \qquad \frac{1}{\rho_0}\,\frac{\partial p}{\partial y} \sim \frac{U^2}{l}\,(\mathrm{Sc}\,\mathrm{Re}^2)^{-1/4};$$

$$\nu\,\frac{\partial^2 v}{\partial x^2} \sim \frac{\nu U}{l^2}\,(\mathrm{Sc}^3\,\mathrm{Re}^2)^{-1/4}; \qquad \nu\,\frac{\partial^2 v}{\partial y^2} \sim \frac{\nu U}{l^2}\,\mathrm{Re}\,(\mathrm{Sc}\,\mathrm{Re}^2)^{-1/4}.$$

In the continuity equation (3), both terms are obviously of the same order, and in the transport equation (4), we obtain the following orders of magnitude:

$$u\,\frac{\partial c}{\partial x}, \quad v\,\frac{\partial c}{\partial y} \sim \frac{Uc_*}{l}\,\mathrm{Sc}^{-1/2}; \quad D\,\frac{\partial^2 c}{\partial x^2} \sim \frac{Dc_*}{l^2}; \quad D\,\frac{\partial^2 c}{\partial y^2} \sim \frac{Dc_*}{l^2}\,(\mathrm{Sc}\,\mathrm{Re}^2)^{1/2}.$$

We divide Eq. (1) by U^2/l, Eq. (2) by $U^2(\mathrm{Sc}\,\mathrm{Re}^2)^{-1/4}/l$, and Eq. (4) by $c_* U\,\mathrm{Sc}^{-1/2}/l$. Passing to the limit $\mathrm{Sc} \to \infty$ and taking into account condition (12) and the definitions of Re and Sc, we obtain the following system of boundary-layer equations:

$$\nu\,\frac{\partial^2 u}{\partial y^2} = g\beta(c - c_0); \tag{13}$$

$$\frac{1}{\rho_0}\,\frac{\partial p}{\partial y} = \nu\,\frac{\partial^2 v}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \tag{14}$$

$$u\,\frac{\partial c}{\partial x} + v\,\frac{\partial c}{\partial y} = D\,\frac{\partial^2 c}{\partial y^2}. \tag{15}$$

Equations of motion in this form have not been considered previously. The form of boundary conditions (6) is unchanged. Of the solutions of problem (13)–(15), (6), we consider only solutions that have finite velocity away from the substrate:

$$\lim_{y\to\infty} u(x, y) = u_\infty(x) < \infty. \tag{16}$$

Here $u_\infty(x)$ is defined in the process of solution of the problem. Problem (13)–(16), (6) describes the motion in a thin dynamic-diffusion layer with thickness of the order of $l(\mathrm{Sc}\,\mathrm{Re}^2)^{-1/4}$, and outside the layer, $c \approx c_0$. In this layer, the buoyancy and viscous forces are of the same order, and the inertial forces and the longitudinal pressure gradient are negligible in comparison with them. In contrast to the case of a classical boundary layer [2], the external representation for velocity is determined from the solution of the problem and not from the matching condition. Obviously, the velocity-vector components and the concentration are obtained separately from the pressure, which is determined by integration of the first of Eqs. (14) over y from y to ∞ taking into account the continuity equation:

$$p(x,y) = p_\infty(x) + \rho_0\nu\Big(u'_\infty(x) - \frac{\partial u}{\partial x}\Big) \tag{17}$$

Here $p_\infty(x)$ is the pressure at the outer edge of the boundary layer.

Since, in the general case, $u_\infty(x) \neq 0$, the solution of problem (13)–(16), (6) cannot be matched with the external solution (10). To resolve this discrepancy, it is necessary to obtain one more asymptotic form of the problem that should describe the motion in a region with asymptotic thickness greater than the boundary-layer thickness considered above. In the limit $\mathrm{Sc} \to \infty$, the following three variants are possible:

$$\mathrm{Sc}/\mathrm{Re}^2 \to 0; \tag{18a}$$

$$\mathrm{Sc}/\mathrm{Re}^2 \sim 1; \tag{18b}$$

$$\mathrm{Sc}/\mathrm{Re}^2 \to \infty. \tag{18c}$$

Let condition (18a) be satisfied. We seek the asymptotic form of system (1)–(4), assuming that

$$x \sim l, \qquad y \sim l(\mathrm{Sc}/\mathrm{Re}^2)^{1/4}, \qquad u \sim U\,\mathrm{Sc}^{-1/2}, \qquad v \sim U(\mathrm{Sc}\,\mathrm{Re}^2)^{-1/4},$$

$$p \sim \rho_0 U^2(\mathrm{Sc}\,\mathrm{Re}^2)^{-1/2}, \qquad c \equiv c_0.$$

Comparison of the orders of magnitude of the quantities in system (1)–(4) shows that Prandtl's hypothesis on the equality of the orders of magnitude of the viscous and inertial forces is valid. Denoting the velocity components by u_1 and v_1, we obtain the system

$$u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y} = \nu\frac{\partial^2 u_1}{\partial y^2}; \tag{19}$$

$$\frac{\partial p}{\partial y} = 0, \qquad \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \tag{20}$$

and, from the matching conditions, we have the boundary conditions

$$u_1\Big|_{y=0} = u_\infty(x), \qquad v_1\Big|_{y=0} = 0, \qquad u_1 \xrightarrow[y\to\infty]{} 0. \tag{21}$$

Problem (19)–(21) describes the motion in a purely dynamic, thin ($c = \mathrm{const}$) boundary layer whose thickness is of the order of $l(\mathrm{Sc}/\mathrm{Re}^2)^{1/4}$, i.e., asymptotically greater than the thickness of a dynamic-diffusion layer. The dynamic layer problem differs from the classical problem in that the longitudinal velocity is specified at the inner rather than at the outer edge. In this problem, the pressure can be considered zero for the following reasons. From the first of Eqs. (20), we find that the pressure p is the same as the pressure at the outer edge of the boundary layer where $p \equiv 0$ (the state of rest; the pressure is equal to hydrostatic pressure). Therefore, in this case, $p_\infty \equiv 0$ in formula (17).

Let condition (18b) be satisfied. We assume that

$$x \sim l, \qquad y \sim l, \qquad u \sim U\,\mathrm{Sc}^{-1/2}, \qquad v \sim U\,\mathrm{Sc}^{-1/2},$$
$$p \sim \rho_0 U^2(\mathrm{Sc}\,\mathrm{Re}^2)^{-1/2}, \qquad c \equiv 0. \tag{22}$$

Comparing the orders of magnitude and taking into account condition (18b), we see that the motion is described by steady Navier–Stokes equations of the form (1)–(3), in which the concentration is constant. The boundary conditions have the form

$$u\Big|_{y=0} = u_\infty(x), \qquad v\Big|_{y=0} = 0, \qquad u \xrightarrow[y\to\infty]{} 0. \tag{23}$$

In this case, the velocity decays at finite distance from the rigid wall.

Finally, we consider condition (18c). Finding an asymptotic form similar to (22), we see that, in this case, the pressure gradient is of the same order as the viscous forces, and the inertial forces are negligible, i.e., the flow is described by Stokes's system with boundary conditions of the form (23).

Thus, to determine the velocity, concentration, and pressure fields, one should first solve the problem for the dynamic-diffusion layer (without the equation for pressure) and calculate the external representation for the velocity $u_\infty(x)$. After that, using this representation as the boundary condition, it is necessary to solve the problem for the external asymptotic form (for the particular case), to determine $p_\infty(x)$, and, finally, to calculate the pressure in the dynamic-diffusion layer from formula (17). The problem for the dynamic-diffusion layer is of greatest interest, since, by solving it, one obtains the velocity and concentration fields near the rigid wall, which is the goal of calculations in most cases.

Let us determine the asymptotic form in the microconvection problem (7)–(9) in the limit $\mathrm{Sc} \to \infty$. Assuming that the viscous and buoyancy forces are of the same order and keeping terms of leading orders in system (7), we obtain the following boundary-layer equations for microconvection:

$$\begin{gathered}\nu(1-\beta(c-c_0))\,\frac{\partial^2 w_1}{\partial y^2} = g\beta(c-c_0), \quad \frac{\partial q}{\partial y} = \nu\,\frac{\partial^2 w_2}{\partial y^2}, \quad \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0, \\ w_1\,\frac{\partial c}{\partial x} + w_2\frac{\partial c}{\partial y} - \beta D\Big(\frac{\partial c}{\partial y}\Big)^2 = D(1-\beta(c-c_0))\,\frac{\partial^2 c}{\partial y^2}.\end{gathered} \tag{24}$$

The boundary conditions have the form

$$\begin{gathered}w_1\Big|_{y=0} = 0, \qquad w_2\Big|_{y=0} = \beta D\,\frac{\partial c}{\partial y}\Big|_{y=0}, \qquad c\Big|_{y=0} = c_*, \\ c\xrightarrow[y\to\infty]{} c_0, \qquad w_1\xrightarrow[y\to\infty]{} w_\infty(x) < \infty,\end{gathered} \tag{25}$$

where the function $w_\infty(x)$ is determined during the solution.

Self-Similar Solutions. Mass-Transfer Formulas. We seek a solution of problem (13)–(16), (6) in the form $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$, and $c = c_0 + (c_0 - c_*)C(\xi)$, where the streamfunction ψ has the form

$$\psi = x^{3/4}\Big(\frac{64 g\beta(c_0 - c_*)D^3}{27\nu}\Big)^{1/4}\Psi(\xi), \qquad \xi = \Big(\frac{3g\beta(c_0-c_*)}{4\nu D}\Big)^{1/4}\frac{y}{x^{1/4}}.$$

Then, Eqs. (13)–(17) become

$$\Psi''' = C, \qquad C'' = -\Psi C'. \tag{26}$$

It follows from condition (6) that

$$\Psi(0) = \Psi'(0) = 0, \qquad C(0) = -1, \qquad C\xrightarrow[\xi\to\infty]{} 0. \tag{27}$$

From condition (16), we find that the external representation of the velocity must satisfy the conditions

$$\lim_{\xi\to\infty}\Psi'(\xi) = U_\infty = \mathrm{const} < \infty, \qquad u_\infty(x) = \sqrt{\frac{4g\beta(c_0-c_*)x}{3\,\mathrm{Sc}}}\,U_\infty. \tag{28}$$

The solutions of problem (26), (27) that do not satisfy condition (28) are not considered since they have no physical meaning.

To characterize the mass transfer between the growing film and the solution, we introduce the overall and local Nusselt numbers:

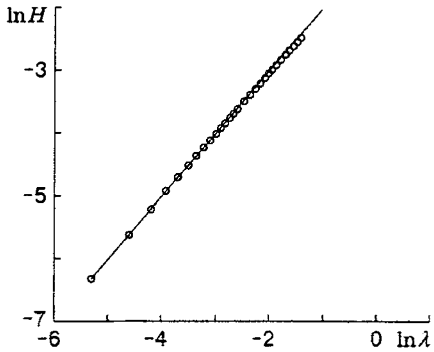

Fig. 1

$$\mathrm{Nu} = \frac{1}{c_0 - c_*} \int_0^l \frac{\partial c}{\partial y}\Big|_{y=0} dx, \qquad \mathrm{Nu}_x = \frac{x}{c_0 - c_*} \frac{\partial c}{\partial y}\Big|_{y=0}.$$

For the solutions of problem (26)–(28), the Nusselt numbers are given by

$$\mathrm{Nu} = (4/3)^{3/4} C'(0)(\mathrm{Sc}\,\mathrm{Re}^2)^{1/4} \approx 0.670(\mathrm{Sc}\,\mathrm{Re}^2)^{1/4},$$
$$\mathrm{Nu}_x = (3/4)^{1/4} C'(0)\left(\mathrm{Sc}\,\mathrm{Re}_x^2\right)^{1/4} \approx 0.502(\mathrm{Sc}\,\mathrm{Re}_x^2)^{1/4}. \tag{29}$$

In this case, problem (26)–(28) was solved numerically. The Reynolds number is given by formula (11), and the local Reynolds number Re_x is defined by formula (11) in which l is replaced by x. Formulas (29) are similar to those obtained in [3], where Nusselt number was derived as a function of the Grashof and Prandtl numbers.

Problem (24), (25) admits self-similarity and, therefore, we seek its solution in the form $w_1 = \partial\psi/\partial y$, $w_2 = -\partial\psi/\partial x$, and $c = c_0 + (c_0 - c_*)C(\xi)$, where the function ψ has the same form as before. Then, for Ψ and C, we obtain the problem

$$(1 - \lambda C)\Psi''' = C, \qquad (1 - \lambda C)C'' = -\lambda(C')^2 - \Psi C',$$
$$\Psi(0) = -\lambda C'(0), \qquad \Psi'(0) = 0, \qquad C(0) = -1, \tag{30}$$
$$C \xrightarrow[\xi\to\infty]{} 0, \qquad \Psi' \xrightarrow[\xi\to\infty]{} U_\infty < \infty,$$

where $\lambda = \beta(c_0 - c_*)$ is the Boussinesq parameter. The formula for the external velocity $w_\infty(x)$ is similar to (28).

Problem (30) was solved numerically, and the value of the parameter λ was varied from 0 to 0.25 [for $\lambda = 0$, this problem coincides with (26)–(28)]. For each λ, the value of Nu was calculated. Numerical results are shown in Fig. 1. The straight line is given by the equation $\ln H = 0.990 \ln\lambda - 1.773$ [$H = (\mathrm{Nu}(0) - \mathrm{Nu}(\lambda))(\mathrm{Re}^2\,\mathrm{Sc})^{-1/4}$], and the points represent numerical values. Insignificant deviations of the numerical points from the straight line appear only at $\lambda \geqslant 0.15$. The physical meaning of λ as the relative deviation from the average density implies that the case of not too large λ is of the greatest importance. Thus, the dependence of the Nusselt number on λ can be written as

$$\mathrm{Nu}(\lambda) = (0.670 - 0.169\lambda^{0.990})(\mathrm{Re}^2\,\mathrm{Sc})^{1/4}. \tag{31}$$

Formulas (29) and (31) coincide at $\lambda = 0$, and dependence (31) is nearly linear for other values of λ.

Let us determine the thickness of the dynamic-diffusion layer. In classical theory, boundary-layer thickness is evaluated using the so-called displacement thickness [2]. In our case, a characteristic feature of the boundary layer is that the concentration c inside the layer differs from the average value, and outside the layer, $c \approx c_0$. An analog of the displacement thickness δ_c^* is defined by the equality

$$\delta_c^*(c_0 - c_*) = \int_0^\infty [c_0 - c(x,y)]\,dy.$$

Calculations for the self-similar solutions yield

$$\delta_c^* = \frac{x}{(\mathrm{Re}_x^2\,\mathrm{Sc})^{1/4}} \int_0^\infty [-C(\xi)]\,d\xi = \frac{x}{(\mathrm{Re}_x^2\,\mathrm{Sc})^{1/4}}\, d(\lambda).$$

In the calculations, the functions $d(\lambda)$ and $U_\infty(\lambda)$ were determined. They turned out to be nearly linear. The following formulas hold:

$$\delta_c^*(x) = (1.165 + 0.260\lambda)\,\frac{x}{(\mathrm{Re}_x^2\,\mathrm{Sc})^{1/4}}, \qquad u_\infty(x) = (0.884 + 0.029\lambda)\sqrt{\frac{4g\lambda x}{3\,\mathrm{Sc}}}. \tag{32}$$

It follows from (32) that the total convection rate, which is determined by U_∞, is almost independent of λ.

Initial Asymptotic Forms. We consider the asymptotic form of problem (1)–(6) in the time interval $[0,\tau]$ as $\tau \to 0$. Then, at small times, the order of magnitude of the velocities is $U_0 = g\beta(c_0 - c_*)\tau D/\nu$. Assuming that $x \sim l$, $y \sim \delta_0 = \sqrt{D\tau}$, $u \sim U_0$, $v \sim \delta_0 U_0/l$, $p \sim \rho_0 \nu U_0/l$, and $c \sim c_*$, we see that the convective terms and the pressure gradient in Eq. (1) are negligible compared to the other terms, i.e., again it is necessary to adopt the assumption that the buoyancy and viscous forces are of the same order of magnitude. Next, in Eq. (2), the higher-order terms are $(1/\rho_0)\partial p/\partial y$ and $\nu \partial^2 v/\partial y^2$, and in Eq. (4), they are $\partial c/\partial t$ and $D\partial^2 c/\partial y^2$. Thus, we have the system of equations

$$\nu \frac{\partial^2 u}{\partial y^2} = g\beta(c - c_0); \tag{33}$$

$$\frac{1}{\rho_0}\frac{\partial p}{\partial y} = \nu \frac{\partial^2 v}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \tag{34}$$

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial y^2}. \tag{35}$$

The initial and boundary conditions for c are given by

$$c\Big|_{t=0} = c_0, \qquad c\Big|_{y=0} = c_*, \qquad c \xrightarrow[y\to\infty]{} c_0. \tag{36}$$

For the velocity components, we specify the attachment condition

$$u\Big|_{y=0} = v\Big|_{y=0} = 0 \tag{37}$$

and the additional condition

$$\lim_{y\to\infty} u(t,x,y) = u_\infty(t,x) < \infty, \tag{38}$$

where $u_\infty(t,x)$ is determined during the solution. Problem (33)–(38) describes the motion in the dynamic-diffusion layer at small times. The admixture concentration c is obtained from Eq. (35) and conditions (36). After determining the concentration, it is possible to find the velocity components and the pressure, and the time appears in the solution as a parameter.

In addition to the asymptotic form considered above, we determine the asymptotic form in the microconvection problem (7)–(9). Assuming that the buoyancy and viscous forces are of the same order of magnitude and retaining high-order terms in system (7)–(9), we obtain the system

$$(1 - \beta(c - c_0))\nu \frac{\partial^2 w_1}{\partial y^2} = g\beta(c - c_0), \quad \frac{\partial q}{\partial y} = \nu \frac{\partial^2 w_2}{\partial y^2}, \quad \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0,$$
$$\frac{\partial c}{\partial t} - \beta D\Big(\frac{\partial c}{\partial y}\Big)^2 = D(1 - \beta(c - c_0))\,\frac{\partial^2 c}{\partial y^2} \tag{39}$$

with the conditions

$$c\Big|_{t=0} = c_0, \quad c\Big|_{y=0} = c_*, \quad c \xrightarrow[y\to\infty]{} c_0,$$
$$w_1\Big|_{y=0} = 0, \quad w_2\Big|_{y=0} = \beta D \frac{\partial c}{\partial y}\Big|_{y=0}, \quad \lim_{y\to\infty} w_1(t,x,y) = w_\infty(t,x) < \infty. \tag{40}$$

Problem (35), (36) represents the well-known problem [2] of the smoothing of an initial temperature jump, whose solution is given by

$$c = c_0 + (c_0 - c_*)\hat{c}(\eta) = c_0 + (c_0 - c_*)\Big(-1 + \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\alpha^2}\, d\alpha\Big), \quad \eta = \frac{y}{2\sqrt{Dt}}.$$

Next, we can seek the velocity components u and v in the form $u = u(t,y)$ and $v \equiv 0$, assuming that at small times they do not depend on the longitudinal coordinate. If we set $u(t,y) = 4Dg\beta(c_0 - c_*)t\hat{u}(\eta)/\nu$, from (33) we obtain the equation $\hat{u}'' = \hat{c}$ or $\hat{u}' = \int_0^\eta \hat{c}(\alpha)\, d\alpha + A$. The additional condition (38) implies that $\hat{u}' \to 0$ as $\eta \to \infty$, and, hence,

$$A = -\int_0^\infty \hat{c}(\alpha)\, d\alpha \quad \text{or} \quad \hat{u}' = -\int_\eta^\infty \hat{c}(\alpha)\, d\alpha.$$

Taking into account (37), we obtain

$$u(t,y) = \frac{4Dg\beta(c_0 - c_*)}{\nu} t\hat{u}(\eta), \qquad \hat{u}(\eta) = -\int_0^\eta \int_\omega^\infty \hat{c}(\alpha)\, d\alpha\, d\omega.$$

From this, for the external representation for the velocity, we have

$$u_\infty(t) = \frac{4Dg\beta(c_0 - c_*)}{\nu} t\hat{U}_\infty, \qquad \hat{U}_\infty = -\int_0^\infty \int_\omega^\infty \hat{c}(\alpha)\, d\alpha\, d\omega.$$

Thus, problem (33)–(38) is solved in quadratures.

In problem (39), (40), we seek self-similar solutions in the form

$$c(t,y) = c_0 + (c_0 - c_*)\hat{c}(\eta), \qquad w_1 = w_1(t,y) = \frac{4Dg\beta(c_0 - c_*)}{\nu} t\hat{w}_1(\eta), \qquad w_2 \equiv 0.$$

For $\hat{w}_1$ and $\hat{c}$, we then obtain the problem

$$(1 - \lambda\hat{c})\hat{w}_1'' = \hat{c}, \qquad (1 - \lambda\hat{c})\hat{c}'' = -\lambda(\hat{c}')^2 - 2\eta\hat{c}',$$
$$\hat{w}_1(0) = 0, \qquad \hat{c}(0) = -1, \qquad \hat{c} \xrightarrow[\eta\to\infty]{} 0, \qquad \hat{w}_1' \xrightarrow[\eta\to\infty]{} 0. \tag{41}$$

The mass transfer is characterized only by the overall Nusselt number because there is no dependence on the x coordinate. A numerical solution of problem (41) yields the following formulas for the mass transfer, the layer thickness, and the velocity in microconvection:

$$\mathrm{Nu}(\lambda) = \Big(\frac{2}{\sqrt{\pi}} - 0.344\lambda^{0.992}\Big)\frac{l}{2\sqrt{Dt}}; \tag{42}$$

$$\delta_c^*(t) = 2(0.606 - 0.055\lambda)\sqrt{Dt}, \qquad u_\infty(t) = (0.249 - 0.117\lambda)4g\lambda t/\mathrm{Sc}. \tag{43}$$

It follows from (31) and (42) that in the steady regime and at small times, the mass-transfer rate (dimensionless) decreases as λ increases. This is due to various physical factors: an increase in the boundary-layer thickness during the stabilization process [see (32)] and a decrease in the convection rate at the beginning of the process [see (43)].

Conclusions. The problem of mass transfer and free convection near a vertical wall at high Schmidt numbers was considered. Integral flow characteristics for the Oberbeck–Boussinesq model and the microconvection model were compared. For both models, asymptotic forms of the problems were derived for the steady flow regime and for small times. In the flow region, a dynamic-diffusion layer is distinguished, in which the buoyancy forces are significant. Outside the layer, the admixture concentration does not differ from the average.

The structure of the velocity field depends on the Reynolds number. If Re is high, in the flow region there is a purely dynamic layer with greater asymptotic thickness, whose inner edge is adjacent to the dynamic-diffusion layer and whose outer edge neighbors the state of rest. For low Re, the Stokes approximation can be used outside the diffusion layer.

For both convection models, we obtained formulas for Nusselt numbers as functions of the Reynolds number, the Schmidt number, and the Boussinesq parameter for both the steady regime and for small times. In the case of convection, these formulas coincide with the well-known formulas obtained previously under the assumption of intense motion at high Re.

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